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Extension of holomorphic vector bundles across a totally real submanifold

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Abstract

Let $\mathcal{E}$ be a Hermitian-holomorphic vector bundle with compatible Chern connection ∇, defined on the complement of $\mathbb{R}^n$ inside a polydisc $\Delta \subseteq \mathbb{C}^n$. The main result of this note provides a necessary and sufficient condition for unique extension of $\mathcal{E}$ to Δ, namely that the curvature F_∇, when contracted with a non-vanishing holomorphic vector field ξ, is $\bar{\partial}$-exact. Applications to removable singularities for anti-self-dual connections across a real line in $\mathbb{C}^2$ and solutions of the static monopole equation across a point in $\mathbb{R}^3$ are also given as corollaries.

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1. Introduction

For every unitary vector bundle on a complex manifold there is a one-to-one correspondence between unitary connections ∇ with curvature form F_∇ of type $(1, 1)$, and compatible holomorphic structures. Since the appearance of the removable singularities theorem of Uhlenbeck [12], criteria for the extendibility of unitary gauge fields across subsets of Euclidean space have evolved from L^2 – "finite energy" – curvature assumptions, but as we hope to indicate in this note, another line of investigation is opened by considering the associated holomorphic structures. In particular, the interplay between unitary connections of curvature type $(1, 1)$ and holomorphic connections will be seen to provide an alternative (though possibly related) class of removable singularities theorems.

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Fundamental to the following discussion is the role played by Hartogs figures in $\mathbb{C}^m$. A typical Hartogs figure H consists of a union of two sets, the first of which corresponds to the cartesian product of a polydisc $D \subseteq \mathbb{C}^{m-1}$ with an annulus $\mathcal{A} \subset \mathbb{C}$, while the second corresponds to the product of an open ball strictly contained in D with a disc which "fills in" the annulus. Their utility in a wide range of extension problems, from continuation of holomorphic and meromorphic functions to extension of analytic subvarieties and coherent analytic sheaves, has been surveyed by Siu [11], drawing together in this work the results of several authors.

The role of holomorphic connections as a means of extending complex vector bundles across compact regions, or analytic subvarieties of codimension two or more, has been examined by Buchdahl and the author in [4], while an application to continuation of unitary gauge fields, and in particular anti-self-dual Yang–Mills fields, has been exposed in [7]. Given a holomorphic vector bundle $\mathcal{E}$ on a Hartogs figure H, the essential idea is first to trivialise $\mathcal{E}$ over the polydisc D. If ξ denotes a non-vanishing holomorphic vector field on H, tangent to the annular fibres $\mathcal{A}$, then existence of a holomorphic Lie derivative L_ξ acting on sections of $\mathcal{E}$ (i.e., a holomorphic relative connection, which provides a canonical "lift" of ξ to the total space of $\mathcal{E}$) determines a trivialisation on H by covariantly constant frames, provided holonomy is trivial. Now the second component of H consists of simply connected discs parametrised by an open ball inside D, and holonomy is parametrised complex-analytically, hence uniqueness of analytic continuation implies that trivial holonomy over the ball extends to D. Unique extension of $\mathcal{E}$ to the smallest polydisc containing $D \times \mathcal{A}$ then follows directly.

Let TX denote the smooth tangent bundle of a complex manifold X. A holomorphic vector bundle $\mathcal{E}$ is said to be "Hermitian-holomorphic" if it is equipped with a Hermitian structure and a uniquely determined unitary connection $\nabla : C^\infty(X, \mathcal{E}) \to C^\infty(X, T^*X \otimes \mathcal{E})$, "compatible" with $\mathcal{E}$ in the following sense. ∇ corresponds to $d + A^{1,0} + A^{0,1}$ in any unitary frame, where the second and third terms are matrix-valued one-forms in $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ respectively, such that $A^{1,0} = -(A^{0,1})^*$ ("$*$" denoting the Hermitian adjoint). Moreover, $\bar\partial A^{0,1} + A^{0,1} \wedge A^{0,1} = 0$ together with the Newlander–Nirenberg theorem for holomorphic vector bundles [2,5] imply that $\bar\partial + A^{0,1}$ locally determines the holomorphic structure $\bar\partial_{\mathcal{E}}$.

Given K a compact subset of a domain $\Omega \subset \mathbb{C}^m$, and $X := \Omega \setminus K$, assume the existence of a "Higgs field" $\psi \in C^\infty(\Omega \setminus K, \mathcal{E} \otimes \mathcal{E}^*)$ [9] such that

$$\bar\partial \psi = F_\nabla \rfloor \xi, \tag{1}$$

the second term denoting contraction of F_∇ by ξ. This was seen in [7], Corollary 3, to provide a sufficient condition for existence of a holomorphic Lie derivative L_ξ, and hence for unique extension of $\mathcal{E}$ across K. In particular, when $K = \{0\}$, Ω corresponds to a ball $B \subseteq \mathbb{R}^4$ and ∇ is an anti-self-dual connection, (1) is effectively equivalent to the "finite energy" condition

$$\|F_\nabla\|^2 := \int_{B\setminus\{0\}} -\mathrm{tr}(F_\nabla \wedge *F_\nabla) < \infty$$

(where $*F_\nabla$ denotes the Hodge dual) used by Uhlenbeck. The L^2-curvature hypothesis has also been applied by Bando [3] to extend Hermitian-holomorphic vector bundles across the origin in $\mathbb{C}^2$.

The concern of the present note, however, is in applying condition (1) to the unique extension of Hermitian-holomorphic vector bundles across totally real submanifolds. In particular, let $\Delta \subseteq \mathbb{C}^{2n-k}$, $k = 0, 1$, $n \geq 2$, represent a polydisc, and $\mathbb{R}^{2n-k}$ denote the set of $(z) \in \mathbb{C}^{2n-k}$ such that the imaginary part $\mathrm{Im}(z_j) = 0$, $1 \leq j \leq 2n-k$. Moreover, define $\mathcal{H}^{1,0}$ to be a sub-bundle of the holomorphic tangent bundle of $\mathbb{C}^{2n-k}$, such that for all $p \in \mathbb{C}^{2n-k}$,

$$\mathcal{H}_p^{1,0} := \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_{2j-1}} + \mathbf{i}\frac{\partial}{\partial z_{2j}},\ 1 \leq j \leq \left[n - \frac{k}{2} \right] \right\},$$

where brackets [ ] here denote the integer part.

Theorem 1. *Consider $\mathcal{E} \to \Delta \setminus \mathbb{R}^{2n-k}$ a Hermitian-holomorphic vector bundle, and ξ a non-vanishing section of $\mathcal{H}^{1,0}$ over Δ. Then there exists $\psi \in C^\infty(\Delta \setminus \mathbb{R}^{2n-k}, \mathcal{E} \otimes \mathcal{E}^*)$ such that*

$$\bar{\partial}\psi = F_\nabla \rfloor \xi$$

if and only if there exists a unique holomorphic extension $\hat{\mathcal{E}} \to \Delta$ such that $\hat{\mathcal{E}} \mid_{\Delta \setminus \mathbb{R}^{2n-k}} \cong \mathcal{E}$.

If $\eta : \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{CP}_m$ denotes the natural quotient map, an example of a holomorphic vector bundle over $\mathbb{C}^m \setminus \mathbb{R}^m$ which admits no holomorphic extension is $\mathcal{E} := \eta^* T^{1,0}\mathbb{CP}_m \mid_{\mathbb{C}^m \setminus \mathbb{R}^m}$, $m = 2n - k \geq 3$. If $\mathcal{E}$ were extendible to $\mathbb{C}^m$ in this case, it would of course be trivial. As in [4], where $\eta^* T^{1,0}\mathbb{CP}_m$ is shown to define a non-trivial bundle over $\mathbb{C}^{m+1} \setminus \{0\}$, the proof of non-triviality rests on the fact that the vanishing locus of a holomorphic function of two or more variables cannot be contained in a totally real subset.

When a Hermitian connection splits into $\nabla = \nabla^{1,0} + \bar{\partial}_{\mathcal{E}}$ such that

$$\bar{\partial}_{\mathcal{E}} \nabla^{1,0} + \nabla^{1,0} \bar{\partial}_{\mathcal{E}} = 0$$

(i.e., the curvature $F_{\nabla^{1,0}}$ has bi-type $(2, 0)$), then $\mathcal{E}$ admits a holomorphic connection. A global corollary of Theorem 1 is the following.

Corollary 1. *Consider M^m a real-analytic manifold, and a complex manifold X^m, $m \geq 3$, corresponding to the complexification of M. Let $\mathcal{E} \to X \setminus M$ be a Hermitian-holomorphic vector bundle. If $\mathcal{E}$ admits a holomorphic connection, then there exists a unique extension $\hat{\mathcal{E}} \to X$. Moreover, if X is a Stein manifold, then $\hat{\mathcal{E}}$ exists if and only if $\mathcal{E}$ admits a holomorphic connection.*

Given M real-analytic, the complexification X can always be constructed as a Stein neighbourhood containing M as the fixed-point set, or "real slice", of an anti-holomorphic involution corresponding locally to complex conjugation on X. In many instances M compact gives rise to a natural compactification of X, as for example $M = \mathbb{S}^4$, $X = \mathbb{G}r(2, 4)$ [6].

A Hermitian-holomorphic vector bundle $\mathcal{E}$ corresponding to an anti-self-dual gauge field on $\mathbb{R}^4 \setminus \mathbb{R}^2$ is necessarily trivial, since F_∇ is of type (1,1) with respect to any metric-compatible complex structure on $\mathbb{R}^4$. In particular, any complex structure for which $\mathbb{R}^4 \setminus \mathbb{R}^2$ corresponds to $\mathbb{C}^2 \setminus \mathbb{C}$ yields a trivialisation of the associated Hermitian-holomorphic vector bundle via the Oka–Grauert principle [8], and consequently the smooth anti-self-dual connection extends uniquely to $\mathbb{R}^4$. Such extensions do not exist in general, however, as a holonomy-classification of singular Sobolev connections, due to Sibner and Sibner [10] has explicitly revealed.

When $\mathcal{E}$ is a Hermitian-holomorphic vector bundle defined on the complement of a totally real line in $\mathbb{C}^2$, the existence of a holomorphic Lie derivative L_ξ, together with the method of continuation by Hartogs figures, is easily seen in Corollary 2 here to generalise the result of [7], corollary 3, as applied to extension of anti-self-dual gauge fields defined initially on the complement of a point in $\mathbb{R}^4$. A further consequence of this result is a removable singularities criterion for solutions of the static Bogomolny equation – viewed as a time-independent reduction of the Yang–Mills equation – defined initially on the complement of a point in $\mathbb{R}^3$.

2. Unique extension of Hermitian-holomorphic bundles

Given a polydisc $\Delta \subseteq \mathbb{C}^{2n-k}$, $n \geq 2$, $k = 0, 1$, recall the definition of $\mathcal{H}^{1,0}$ as a sub-bundle of the holomorphic tangent bundle of $\mathbb{C}^{2n-k}$, such that for all $p \in \mathbb{C}^{2n-k}$,

$$\mathcal{H}^{1,0}_p := \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_{2j-1}} + \mathbf{i}\frac{\partial}{\partial z_{2j}},\ 1 \leq j \leq \left[n - \frac{k}{2}\right] \right\}.$$

Theorem 2. *Consider $\mathcal{E} \to \Delta \setminus \mathbb{R}^{2n-k}$ a Hermitian-holomorphic vector bundle, and ξ a non-vanishing section of $\mathcal{H}^{1,0}$* over Δ. *Then there exists $\psi \in C^\infty(\Delta \setminus \mathbb{R}^{2n-k}, \mathcal{E} \otimes \mathcal{E}^*)$ such that*

$$\bar{\partial}\psi = F_\nabla \rfloor \xi$$

if and only if there exists a unique holomorphic extension $\hat{\mathcal{E}} \to \Delta$ such that $\hat{\mathcal{E}}\,|_{\Delta \setminus \mathbb{R}^{2n-k}} \cong \mathcal{E}$.

Proof. The argument will be simplified slightly if the following $\mathbb{C}$-linear change of coordinates is imposed on $\mathbb{C}^{2n-k}$. Let $w_{2j-1} := z_{2j-1}$, $w_{2j} := z_{2j-1} + \mathbf{i}z_{2j}$, $1 \leq j \leq [n - k/2]$, and $w_{2n-1} := z_{2n-1}$. Now consider $\varpi : \mathbb{C}^{2n-k} \to \mathbb{C}^{n-k}$, where $\varpi(w) = (\tilde{w})$ such that $\tilde{w}_j := w_{2j}$, $1 \leq j \leq [n - \frac{k}{2}]$, and without loss of generality, assume $\xi = \partial/\partial w_1$. Let $\mathfrak{R}$ denote the image of $\mathbb{R}^{2n-k}$ under the above coordinate transformation, determined explicitly by the equations $\mathrm{Im}(w_{2j-1}) = 0, \mathrm{Re}(w_{2j-1}) = \mathrm{Re}(w_{2j})$, $1 \leq j \leq [n - k/2]$, and $\mathrm{Im}(w_{2n-1}) = 0$. Consequently $\varpi^{-1}(\tilde{w})$ determines an n-dimensional complex subspace of $\mathbb{C}^{2n-k}$ such that for all $\tilde{w} \in \varpi(\Delta)$, $k = 0$ implies $\mathfrak{R} \cap \varpi^{-1}(\tilde{w})$ consists of a unique (w) such that $w_{2j-1} = \mathrm{Re}(\tilde{w}_j)$, $w_{2j} = \tilde{w}_j$, $1 \leq j \leq n$. Alternatively, $k = 1$ implies $\mathfrak{R} \cap \varpi^{-1}(\tilde{w}) = (w)$ such that $w_{2j-1} = \mathrm{Re}(\tilde{w}_j)$, $w_{2j} = \tilde{w}_j$, $1 \leq j \leq n - 1$, and $\mathrm{Im}(w_{2n-1}) = 0$. Choose $p \in \mathfrak{R} \cap \Delta$, and for $\varepsilon \in \mathbb{R}_+$, define

$$\mathcal{N}(p,\varepsilon) := \{(w) \in \Delta \,|\mathrm{Re}(w_2) \in (\mathrm{Re}(w_2(p)) - \varepsilon, \mathrm{Re}(w_2(p)) + \varepsilon)\}.$$

Moreover, for $R > \varepsilon$, let

$$T_p(R,\varepsilon) := \{(w) \in \mathcal{N}(p,\varepsilon) \mid |w_1 - \mathrm{Re}(w_2(p))| \leq R\}$$

so that

$$\{(w) \in \mathcal{N}(p,\varepsilon) \mid |w_1 - \mathrm{Re}(w_2)| < R - \varepsilon\} \subset T_p(R,\varepsilon).$$

Let $\pi : \mathbb{C}^{2n-k} \to \mathbb{C}^{2n-k-1}$ be the projection which deletes w_1. Then $\pi(\mathcal{N}(p,\varepsilon))$ is a $2n-k-1$-dimensional polydisc, and $\mathcal{N}(p,\varepsilon) \setminus \mathfrak{R}$ contains

$$\mathcal{N}(p,\varepsilon) \setminus T_p(R,\varepsilon) \cong \pi(\mathcal{N}(p,\varepsilon)) \times \{w_1 \in \mathbb{C} \mid R < |w_1 - \mathrm{Re}(w_2(p))| < S\}.$$

Now let

$$D_p := \{(w) \in \mathcal{N}(p,\varepsilon) \setminus T_p(R,\varepsilon) \mid w_1 = c\} \cong \pi(\mathcal{N}(p,\varepsilon)) \subset \mathbb{C}^{2n-k-1},$$

and observe that $\mathcal{E}\,|_{D_p} \cong \mathcal{O}^r_{\mathbb{C}^{2n-k-1}}$ by the Oka–Grauert Principle. The construction of an appropriate Hartogs figure is carried out in each of two cases as follows.

Case 1: $k = 0$. Choose q in $\pi(\mathcal{N}(p,\varepsilon))$ minus the set of $\pi(w)$ such that $w_{2j-1} = Re(w_{2j})$, $2 \leq j \leq n$, and a sufficiently small open neighbourhood B_q such that $\pi^{-1}(B_q) \cap \mathfrak{R} = \emptyset$. Now $\hat{B}_q := \pi^{-1}(B_q) \cap \mathcal{N}(p,\varepsilon)$ has $\hat{B}_q \cap D_p \cong B_q$, hence define $H_p := (\mathcal{N}(p,\varepsilon) \setminus T_p(R,\varepsilon)) \cup \hat{B}_q$.

Case 2: $k = 1$ Choose q in $\pi(\mathcal{N}(p,\varepsilon))$ minus the set of $\pi(w)$ such that $\mathrm{Im}(w_{2n-1}) = 0$, and a sufficiently small open neighbourhood B_q such that $\pi^{-1}(B_q) \cap \mathfrak{R} = \emptyset$. Hence define H_p as above.

The open sets $\mathcal{N}(p,\varepsilon) \setminus T_p(R,\varepsilon)$ and $\hat{B}_q$ have the property that, for ε and R sufficiently small, holomorphic frames $\mathbf{f}_1$, $\mathbf{f}_2$ of $\mathcal{E}$ may be defined on each set, once again as a consequence of the Oka–Grauert Principle. Relative to local complex coordinates on Δ, let ξ correspond to $\partial/\partial w_1$. Recall that F_∇ is of type (1,1), such that with respect to any ∇-compatible holomorphic frame of $\mathcal{E}$, $A^{0,1} = \mathbf{0}$. Locally,

$$F_\nabla \rfloor \xi = \Sigma_\nu \frac{\partial A_1^{1,0}}{\partial \bar{w}_\nu}\, \mathrm{d}\bar{w}_\nu$$

so that $F_\nabla \rfloor \xi = \bar{\partial}\psi$ implies that $A_1^{1,0} - \psi$ is a holomorphic matrix. Now if $\mathbf{f}_{D_p}$ denotes a holomorphic frame of $\mathcal{E}$ restricted to D_p, a new holomorphic frame $\mathbf{f}_\pi$ may be constructed by "parallel propagation" of $\mathbf{f}_{D_p}$ along the fibres of π. With respect to $\nabla_1^{1,0} = \partial/\partial w_1 + A_1^{1,0} - \psi$, such a "covariantly constant" frame on $\mathcal{N}(p,\varepsilon) \setminus T_p(R,\varepsilon)$ satisfies the ordinary differential equation and initial data

$$\frac{\partial \mathbf{f}_\pi}{\partial w_1} + (A_1^{1,0} - \psi)\mathbf{f}_\pi = \mathbf{0}, \qquad \mathbf{f}_\pi(p') = \mathbf{f}_{D_p}(p') \tag{2}$$

for all $p' \in D_p$. Since $\pi^{-1}(\pi(p')) \cap (\mathcal{N}(p,\varepsilon) \setminus T_p(R,\varepsilon))$ is an annulus, it follows that a well-defined solution of (2) may be obstructed on each fibre by holonomy, i.e., an automorphism

$\alpha \in \mathbf{GL}(r, \mathbb{C})$ such that for a circular path $\gamma_{p'} \subset \pi^{-1}(\pi(p')) \cap (\mathcal{N}(p, \varepsilon) \setminus T_p(R, \varepsilon))$, where $\gamma_{p'}(0) = \gamma_{p'}(1) = p'$, we have $\mathbf{f}_\pi(\gamma_{p'}(1)) = \alpha \cdot \mathbf{f}_\pi(\gamma_{p'}(0))$. Now let $\tilde{\mathbf{f}}_{D_p} := g \cdot \mathbf{f}_{D_p}$ denote a holomorphic gauge-equivalent frame on $\hat{B}_q \cap D_p$. Then a simple gauge transformation of (2) indicates that $\tilde{\mathbf{f}}_\pi := \tilde{g} \cdot \mathbf{f}_\pi$ (with respect to an extended gauge transformation $\tilde{g}$ on $(\mathcal{N}(p, \varepsilon) \setminus T_p(R, \varepsilon)) \cap \hat{B}_q)$ satisfies

$$\frac{\partial \tilde{\mathbf{f}}_\pi}{\partial w_1} + \tilde{A}_1^{1,0} \tilde{\mathbf{f}}_\pi = \mathbf{0}, \quad \tilde{\mathbf{f}}_\pi(p') = \tilde{\mathbf{f}}_{D_p}(p') \tag{3}$$

for all $p' \in D_p \cap \hat{B}_q$, where

$$\tilde{A}_1^{1,0} = \tilde{g}^{-1} \cdot (A_1^{1,0} - \psi) \cdot \tilde{g} + g^{-1} \cdot \frac{\partial \tilde{g}}{\partial w_1}.$$

Given that $\pi^{-1}(\pi(p')) \cap \hat{B}_q$ is simply connected, however, it follows that holonomy associated with solutions of (3) must be trivial, and since it varies holomorphically along D_p, the holonomy α associated with (2) must also be trivial by uniqueness of analytic continuation. $\mathbf{f}_\pi$ thus extends to a covariantly constant holomorphic frame on H_p, and hence on $\mathcal{N}(p, \varepsilon) \setminus \mathfrak{R}$ by homotopy equivalence of paths. If a constant frame corresponding to the standard basis of $\underline{\mathbb{C}}^r$ is now chosen on $\mathcal{N}(p, \varepsilon)$, then the holomorphic gauge transformation $g := \mathbf{f}_\pi$ extended to $\mathcal{N}(p, \varepsilon) \setminus \mathfrak{R}$ yields this constant frame as a local extension of $\mathcal{E}$, with $\partial/\partial w_1$ as the local representation of $\nabla_1^{1,0}$. Clearly the entire procedure may be carried out for any point $p \in \Delta \cap \mathfrak{R}$. Hence $\mathcal{E}$ extends to Δ, and uniqueness follows automatically from the fact that all holomorphic vector bundles on Δ are trivial. Conversely, given the trivial extension $\hat{\mathcal{E}} \to \Delta$, $\bar{\partial}(F_\nabla \rfloor \xi) = 0$ automatically implies $F_\nabla \rfloor \xi$ is $\bar{\partial}$-exact by the Dolbeault lemma.

Remark 1. It is interesting to compare the extension method above with one commonly employed to extend holomorphic functions, initially defined on $\mathbb{C}^m \setminus \mathbb{R}^m$ for all $m \geq 2$. In particular, when $m = 2$, a biholomorphic coordinate transformation of the form $(w_1, w_2) = (z_1 + \mathbf{i}z_2^2, z_2)$ allows the image of $\mathbb{R}^2$ to be spanned locally by a standard Hartogs figure, and hence a holomorphic function f, composed with the above transformation, extends uniquely across the image. Each annular fibre of the Hartogs figure encompasses either two points of the image of $\mathbb{R}^2$ or none, depending on whether the imaginary part of the fibre coordinate is non-negative or negative. Accordingly, the homotopy of loops associated with each fibre is either freely generated by two distinct classes, or trivial. Note that while this has no effect on the definition of an extension $\hat{f}$ of f, it clearly obstructs the trivialisation of a holomorphic vector bundle on the Hartogs figure, since parallel transport of frames need not be well-defined.

Corollary 2. *Consider M^m a real-analytic manifold, and a complex manifold X^m, $m \geq 3$, corresponding to the complexification of M. Let $\mathcal{E} \to X \setminus M$ be a Hermitian-holomorphic vector bundle. If $\mathcal{E}$ admits a holomorphic connection, then there exists a unique extension $\hat{\mathcal{E}} \to X$. Moreover, if X is a Stein manifold, then $\hat{\mathcal{E}}$ exists if and only if $\mathcal{E}$ admits a holomorphic connection.*

Proof. For every Hermitian-holomorphic vector bundle, the curvature F_∇ of the compatible Hermitian connection determines a cohomology class

$$\omega_{\mathcal{E}} \in H^1(X \setminus M, \mathcal{O}(T^{1,0}X)^* \otimes \mathcal{E} \otimes \mathcal{E}^*),$$

where $(T^{1,0}X)^*$ denotes the holomorphic cotangent bundle. The class $\omega_{\mathcal{E}}$ defines the unique obstruction to existence of a holomorphic connection [1], hence $\mathcal{E}$ admits a holomorphic connection precisely when there exists $\varphi \in C^\infty(X \setminus M, (T^{1,0}X)^* \otimes \mathcal{E} \otimes \mathcal{E}^*)$ such that $\bar{\partial}\varphi = F_\nabla$. In particular, note that $F_{\nabla-\varphi}$ has curvature of type (2,0), hence $\nabla - \varphi$ is the required holomorphic connection. Now choose $p \in M$ and a sufficiently small coordinate neighbourhood $\mathcal{U}_p$, such that $M \cap \mathcal{U}_p$ is biholomorphically equivalent to $\mathbb{R}^n \cap \Delta$ above. For any non-vanishing holomorphic vector field ξ corresponding to a section of $\mathcal{H}^{1,0}$ on $\mathcal{U}_p$, let $\psi := \varphi \rfloor \xi$, so that the equation $\bar{\partial}\psi = F_\nabla \rfloor \xi$ is satisfied on $\mathcal{U}_p$. Existence of $\hat{\mathcal{E}}$ then follows directly from the preceding theorem. To verify uniqueness, let $\mathcal{E}_1$, $\mathcal{E}_2 \to X$ denote vector bundle extensions obtained by the above process, i.e., $\mathcal{E}_1 |_{X\setminus M} \cong \mathcal{E}_2 |_{X \setminus M} \cong \mathcal{E}$. Let $\Phi \in \Gamma(X \setminus M, \mathcal{E}_2 \otimes \mathcal{E}_1^*)$ denote the isomorphism between these holomorphic bundles when restricted to the complement of M, hence $\det(\Phi)_p \neq 0$ for all $p \in X \setminus M$. In a similar manner to the above, Φ may be extended uniquely to X via Hartogs' theorem, hence if it is non-empty, the vanishing locus of $\det(\Phi)$ is now supported by an analytic hypersurface in X, which must intersect $X \setminus M$. However, this clearly contradicts the fact that $\Phi |_{X \setminus M}$ is an isomorphism, hence $\mathcal{E}_1 \cong \mathcal{E}_2 \cong \hat{\mathcal{E}}$. It remains to observe that the converse statement follows from Cartan's Theorem B when X is a Stein manifold, since the cohomology obstruction to existence of a holomorphic connection on $\hat{\mathcal{E}} \to X$ vanishes. □

Corollary 3. *Let L be a real line in $\mathbb{R}^4$, and $\mathcal{E} \to B \setminus L$ a Hermitian-holomorphic vector bundle over a ball minus L, with a compatible connection ∇. Suppose ξ is a non-vanishing holomorphic vector field on $\mathbb{C}^2$ such that the flow of ξ intersects each point of L transversely. If there exists $\psi \in C^\infty(B \setminus L, \mathcal{E} \otimes \mathcal{E}^*)$ such that $\bar{\partial}\psi = F_\nabla \rfloor \xi$, then there exists a unique Hermitian potential A over the ball, which is gauge-equivalent to the potential associated with ∇ over $B \setminus L$.*

Proof. Consider $\pi : \mathbb{C}^2 \to \mathbb{C}$ the projection which deletes z_1. Note that $\pi^{-1}(z_2) \cap L = \emptyset$ for all $z_2 \in \mathbb{C} \setminus \pi(L)$, otherwise the intersection consists of a single point. If $L_{\mathbb{C}}$ denotes the complexification of L as a complex subspace of $\mathbb{C}^2$, for all $p \in B$ let $\lambda(p) := \pi^{-1}(\pi(p)) \cap L_{\mathbb{C}}$. Now choose $p \in L \cap B$ and sufficiently small numbers $S > \varepsilon \in \mathbb{R}_+$ such that

$$\mathcal{N}(p, \varepsilon) := \{p' \in B \mid |z_1(\lambda(p')) - z_1(p)| < \varepsilon,\ |z_1(p') - z_1(p)| < S\}.$$

Moreover for $S > R > \varepsilon$, let

$$T_p(R, \varepsilon) := \{p' \in \mathcal{N}(p, \varepsilon) \mid |z_1(p') - z_1(p)| \le R\},$$

so that

$$\{p' \in \mathcal{N}(p, \varepsilon) \mid |z_1(p') - z_1(\lambda(p'))| < R - \varepsilon\} \subset T_p(R, \varepsilon).$$

Note $\pi(\mathcal{N}(p,\varepsilon))$ is a disc in $\mathbb{C}$, and $\mathcal{N}(p,\varepsilon)\setminus L$ contains

$$\mathcal{N}(p,\varepsilon)\setminus T_p(R,\varepsilon) \cong \pi(\mathcal{N}(p,\varepsilon)) \times \{z_1 \in \mathbb{C} \mid R < |z_1 - z_1(p)| < S\}.$$

Now let

$$D_p := \{p' \in \mathcal{N}(p,\varepsilon)\setminus T_p(R,\varepsilon) \mid z_1 = c\} \cong \pi(\mathcal{N}(p,\varepsilon)) \subset \mathbb{C},$$

and observe that $\mathcal{E}\mid_{D_p} \cong \mathcal{O}^r_{\mathbb{C}}$ by the Oka–Grauert Principle. Choose q and a sufficiently small open neighbourhood B_q in $\pi(\mathcal{N}(p,\varepsilon)\setminus L)$, such that $\pi^{-1}(B_q)\cap L = \emptyset$. Now $\hat{B}_q := \pi^{-1}(B_q)\cap\mathcal{N}(p,\varepsilon)$ has $\hat{B}_q \cap D_p \cong B_q$, hence define a Hartogs figure $H_p := (\mathcal{N}(p,\varepsilon)\setminus T_p(R,\varepsilon)) \cup \hat{B}_q$. Now, without loss of generality, assume $\xi := \partial/\partial z_1$ on a sufficiently small neighbourhood of p. The remainder of the argument proceeds as in Theorem 2 above. □

Remark 2. Note that a special case of the above corresponds to ∇ an anti-self-dual connection with respect to the associated unitary structure.

As a final corollary to the main theorem, consider B a ball in $\mathbb{R}^3$, and $E \to B\setminus\{0\}$ a unitary vector bundle, with Hermitian connection ∇ represented locally by a potential $A = A_1\,\mathrm{d}x_1 + A_2\,\mathrm{d}x_2 + A_3\,\mathrm{d}x_3$. The *static Bogomolny equation* for monopoles is a field equation on $\mathbb{R}^3$ which may be realised as a time-independent reduction of the anti-self-dual Yang–Mills equation on $\mathbb{R}^4$. In particular, if $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ simply deletes the t-coordinate, then

$$\pi^* F_\nabla + \pi^*(*F_\nabla) \wedge \mathrm{d}t \tag{4}$$

naturally defines an anti-self-dual two-form on $\mathbb{R}^4$ (note that the Hodge star in (4) refers to forms on $\mathbb{R}^3$). A Hermitian connection ∇' on $\pi^*\mathcal{E}$ for which (4) corresponds to $F_{\nabla'}$ can be constructed in each local frame by defining $A' := A + \varphi\,\mathrm{d}t$ such that the (Bogomolny) equation

$$F_\nabla = 2 * (\mathrm{d}\varphi + [A,\varphi])$$

is satisfied. Here φ represents the "electrostatic potential" of the monopole field (cf. [9]). Recall that ∇' is compatible with the holomorphic structure of $\pi^*\mathcal{E}$ over $\pi^*B \subseteq \mathbb{C}^2$, hence $\overline{\partial}_{\pi^*\mathcal{E}}$ is represented locally by matrices of the form

$$A^{0,1} := \tfrac{1}{2}(A_1 + \mathbf{i}A_2)\,\mathrm{d}\bar{z}_1 + \frac{1}{2}(A_3 + \mathbf{i}\varphi)\,\mathrm{d}\bar{z}_2,$$

with respect to complex coordinates $z_1 = x_1 + \mathbf{i}x_2,\ z_2 = x_3 + \mathbf{i}t$. Define an operator

$$D : C^\infty(\mathcal{E}) \to C^\infty((\mathbb{C}\otimes T\mathbb{R}^3)^* \otimes \mathcal{E})$$

such that for any section σ of $\mathcal{E}$,

$$D(\sigma) := \overline{\partial}_{\pi^*\mathcal{E}}\pi^*(\sigma)\mid_{\mathbb{C}\otimes T\mathbb{R}^3}.$$

As a slight abuse of notation we shall use "D" also to denote the induced operator on $\mathcal{E} \otimes \mathcal{E}^*$.

Corollary 4. *Let $F_\nabla = \Sigma_{\mu,\nu} F_{\mu,\nu}\, \mathrm{d}x_\mu \wedge \mathrm{d}x_\nu$ represent a static monopole field on $B \setminus \{0\} \subseteq \mathbb{R}^3$. If there exists $\psi \in C^\infty(B \setminus \{0\}, \mathcal{E} \otimes \mathcal{E}^*)$ such that*

$$D\psi = F_\nabla \rfloor \frac{\partial}{\partial z_1},$$

then there exists a unique connection over B which is gauge-equivalent to ∇ over $B \setminus \{0\}$.

Proof. Note

$$F_{\nabla'} \rfloor \frac{\partial}{\partial z_1} = \pi^* \left(F_\nabla \rfloor \frac{\partial}{\partial z_1} \right) - \pi^* \left(*F_\nabla \rfloor \frac{\partial}{\partial z_1} \right) \mathrm{d}t \tag{5}$$

while it is easily checked that

$$F_\nabla \rfloor \frac{\partial}{\partial z_1} = -\mathbf{i} \left(F_{1,2}\, \mathrm{d}\bar{z}_1 + \left(*F_\nabla \rfloor \frac{\partial}{\partial z_1} \right) \mathrm{d}x_3 \right) \tag{6}$$

Moreover, $\psi \in C^\infty(B \setminus \{0\}, \mathcal{E} \otimes \mathcal{E}^*)$ implies that in any local frame

$$\bar{\partial}_{\pi^*\mathcal{E}} \pi^*(\psi) = \pi^*(D\psi) - \mathbf{i} \left(\frac{\partial}{\partial x_3} + \frac{1}{2} (A_3 + \mathbf{i}\varphi) \right) \pi^*(\psi)\, \mathrm{d}t \tag{7}$$

But $D\psi = F_\nabla \rfloor \partial/\partial z_1$, together with (6), implies

$$\frac{\partial \psi}{\partial x_3} + \frac{1}{2}(A_3 + \mathbf{i}\varphi)\psi = -\mathbf{i} \left(*F_\nabla \rfloor \frac{\partial}{\partial z_1} \right),$$

hence from (5) and (7) it follows that $\bar{\partial}_{\pi^*\mathcal{E}} \pi^*(\psi) = F_{\nabla'} \rfloor \frac{\partial}{\partial z_1}$. Now consider any ball $B' \subseteq \mathbb{R}^4$ such that $B' \cap \mathbb{R}^3 = B$, let L correspond to the t-axis, and apply the previous corollary. The unique anti-self-dual connection obtained on B' is gauge-equivalent to ∇', hence time-independent, and satisfies the Bogomolny equation on B. □

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